MA114 Summer 2018
Worksheet 16 – Average Value and Volumes I – 7/10/18

1. Find the average value of the following functions over the given interval.

a) $f(x) = x^3$, $[0, 4]$

$$f_{avg} = \frac{1}{4-0}\int_0^4 x^3\,dx = \frac{1}{4}\cdot\frac{1}{4}x^4\Big|_0^4 = \frac{4^4}{4^2} = 16$$

b) $f(x) = x^3, [-1, 1]$

$$f_{avg} = \frac{1}{1-(-1)}\int_{-1}^{1} x^3\,dx = \frac{1}{2}\left(\frac{1}{4}x^4\right)\Big|_{-1}^{1} = \frac{1}{8} - \frac{1}{8} = 0$$

c) $f(x) = e^{-nx}, [-1, 1]$

$$f_{avg} = \frac{1}{1-(-1)}\int_{-1}^{1} e^{-nx}\,dx = \frac{1}{2}\left(-\frac{1}{n}e^{-nx}\right)\Big|_{-1}^{1}$$

$$= \frac{-1}{2n}\left(e^{-1} - e^{1}\right)$$

$$= \frac{e^{1} - e^{-1}}{2n} = \frac{1}{n}\sinh(1)$$

d) $f(x) = \dfrac{1}{1+x^2}, [-1, 1]$

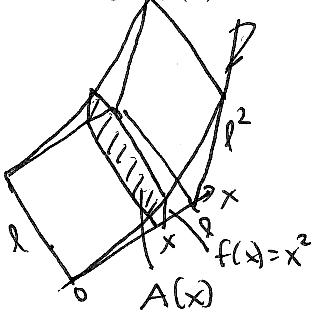
$\arctan(1) = \frac{\pi}{4}$ b/c $\tan(\frac{\pi}{4}) = 1$

$$f_{avg} = \frac{1}{1-(-1)}\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{1}{2}\arctan(x)\Big|_{-1}^{1} = \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

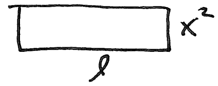
$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

2. Calculate the volume of the following solid. The base is a square, one of whose sides is the interval $[0, l]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.



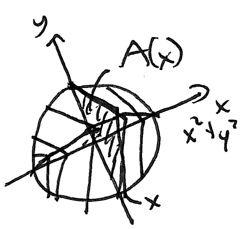
The area $A(x)$ of a cross-section is

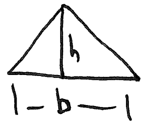


$$A(x) = l x^2$$

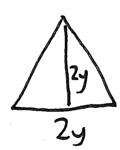
$$\text{So } V = \int_0^l l x^2\,dx = \frac{1}{3} l x^3 \Big|_0^l = \boxed{\frac{1}{3} l^4}$$

3. Calculate the volume of the solid whose base is the unit circle $x^2 + y^2 = 1$ and whose cross sections perpendicular to the x-axis are triangle where the height and base are equal.

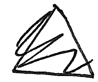




$b = h$, $\frac{b}{2} = y$, where $x^2+y^2=1$, so $y = \sqrt{1-x^2}$



$\to$



$$A(x) = \frac{1}{2}(2y)(2y) = 2y^2 = 2(1-x^2)$$

$$\text{So } V = \int_{-1}^{1} 2(1-x^2)\,dx = 2\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 2\left(1-\frac{1}{3}\right) - \left\{2\left(-1+\frac{1}{3}\right)\right\}$$

$$= \frac{4}{3} + \frac{4}{3}$$

$$= \boxed{\frac{8}{3}}$$